

HAMILTONIZATION AND INTEGRABILITY OF THE CHAPLYGIN SPHERE IN \mathbb{R}^n

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ABSTRACT. The paper studies a natural n -dimensional generalization of the classical nonholonomic Chaplygin sphere problem. We prove that for a specific choice of the inertia operator, the restriction of the generalized problem onto zero value of the $SO(n-1)$ -momentum mapping becomes an integrable Hamiltonian system after an appropriate time reparametrization.

1. INTRODUCTION

Nonholonomic systems are not Hamiltonian. Apparently, Chaplygin was one of the first who considered a time reparametrization in order to transform nonholonomic systems to the Hamiltonian form [9]. Also, after [8], one of the most famous solvable problems in nonholonomic mechanics, describing the rolling without slipping of a balanced ball over a horizontal surface, is referred as the *Chaplygin sphere*, see [1, 19]. It is interesting that the Hamiltonization of the system by the use of a time reparametrization was done just recently by Borisov and Mamaev [4, 5] (for a geometrical setting within a framework of almost Poisson brackets, see [21]).

Fedorov and Kozlov constructed natural n -dimensional model of the Chaplygin-sphere problem and found an invariant measure [14]. Various aspects of the problem are studied in [27, 18, 22]. In [22], it is proved that the reduced equations of motion of the homogeneous ball are already Hamiltonian. However, the general problem of integrability and Hamiltonization is still unsolved.

1.1. Natural Nonholonomic Systems. Let Q be a n -dimensional Riemannian manifold with a nondegenerate metric $\kappa(\cdot, \cdot)$, $V : Q \rightarrow \mathbb{R}$ be a smooth function and let \mathcal{D} be a nonintegrable $(n-k)$ -dimensional distribution of the tangent bundle TQ . A smooth path $q(t) \in Q$, $t \in \Delta$ is called *admissible* (or allowed by constraints) if the velocity $\dot{q}(t)$ belongs to $\mathcal{D}_{q(t)}$ for all $t \in \Delta$. Let $q = (q_1, \dots, q_n)$ be some local coordinates on Q in which the constraints are written in the form

$$(1) \quad (\alpha_q^j, \dot{q}) = \sum_{i=1}^n \alpha_i^j \dot{q}_i = 0, \quad j = 1, \dots, k,$$

where α^j are independent 1-forms. The admissible path $q(t)$ is a *motion of the natural mechanical nonholonomic system* $(Q, \kappa, V, \mathcal{D})$ (or a *nonholonomic geodesic* for $V \equiv 0$) if it satisfies the Lagrange-d'Alembert equations

$$(2) \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial L}{\partial q_i} = \sum_{j=1}^k \lambda_j \alpha^j(q)_i, \quad i = 1, \dots, n.$$

Here the Lagrange multipliers λ_j are chosen such that the solutions $q(t)$ satisfy constraints (1) and the Lagrangian is given by the difference of the kinetic and potential energy: $L(q, \dot{q}) = \frac{1}{2} \sum_{ij} \kappa_{ij} \dot{q}_i \dot{q}_j - V(q)$. The expression $\sum_{j=1}^k \lambda_j \alpha^j(q)_i$ represents the *reaction forces* of the constraints (1).

Applying the Legendre transformation $p_i = \partial L / \partial \dot{q}_i = \sum_j \kappa_{ij} \dot{q}_j$ one can also write the Lagrange-d'Alembert equations as a first-order system on the submanifold $\mathcal{M} = \kappa(\mathcal{D})$ of the cotangent bundle T^*Q :

$$(3) \quad \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} + \sum_{j=1}^k \lambda_j \alpha^j(q)_i, \quad i = 1, \dots, n,$$

where the Hamiltonian is $H(q, p) = \frac{1}{2} \sum_{ij} \kappa^{ij} p_i p_j + V(q)$. As for Hamiltonian systems, it is a first integral of the system.

1.2. Symmetries, Chaplygin Reduction and Hamiltonization. Suppose that a Lie group K acts by isometries on (Q, κ) preserving the potential function V (the Lagrangian L is K -invariant) and let ξ_Q be the vector field on Q associated to the action of one-parameter subgroup $\exp(t\xi)$, $\xi \in \mathfrak{k} = \text{Lie}(K)$. The following version of the *Noether theorem* holds (see [1, 3]): if ξ_Q is a section of the distribution \mathcal{D} then

$$(4) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}}, \xi_Q \right) = \frac{d}{dt} (p, \xi_Q) = 0.$$

In other words, if $\Phi_K : T^*Q \rightarrow \mathfrak{k}^*$ is the momentum mapping of the K -action with respect to the canonical symplectic structure on T^*Q , then $\Phi_K(\xi)$ is conserved along the flow of (3).

On the other side, suppose that Q has a principal bundle structure $\pi : Q \rightarrow Q/K$ and that \mathcal{D} is a K -invariant collection of horizontal spaces of a principal connection,

$$(5) \quad T_q Q = \mathcal{D}_q \oplus \mathfrak{k}_q, \quad \mathfrak{k}_q = \{ \xi_Q(q) | \xi \in \mathfrak{k} \}, \quad q \in Q.$$

Then $(Q, \kappa, V, \mathcal{D})$ is called a *K-Chaplygin system*. The system (2) is K -invariant and reduces to the tangent bundle $T(Q/K) \cong \mathcal{D}/K$ with the reduced Lagrangian L_{red} induced from $L|_{\mathcal{D}}$.

Let H_{red} be a natural mechanical Hamiltonian, the Legendre transformation of L_{red} . The reduced vector field X_{red} on the cotangent bundle $T^*(Q/K)$ can be written in the almost Hamiltonian form

$$(6) \quad i_{X_{red}}(\Omega + \Xi) = dH_{red},$$

where Ω is the canonical symplectic form on $T^*(Q/K)$, Ξ is a semi-basic form depending of the momentum mapping Φ_K and the curvature of the connection \mathcal{D} (for the details see [24, 3, 7, 29]).

In some cases the equations (2), i.e, (3) have a rather strong property - an invariant measure (e.g, see [1, 31]). Within the class of K -Chaplygin systems, the existence of an invariant measure is closely related with their reduction to a Hamiltonian form.

Suppose that the form $\Omega + \Xi$ is conformally symplectic $d(\mathcal{N}(\Omega + \Xi)) = 0$ (it is assumed that \mathcal{N} is a function on Q/K). In this case the system (6) has an invariant measure $\mathcal{N}^{d-1}\Omega^d$, $d = \dim(Q/K)$ and after a time rescaling $d\tau = \mathcal{N}dt$ it becomes the Hamiltonian system with respect to the form $\mathcal{N}(\Omega + \Xi)$. For $d = 2$ the above statement can be inverted: an existence of an invariant measure implies that the nonholonomic form $\omega + \Xi$ is conformally symplectic, see [9, 28, 16, 7, 29, 11]. The conformal factor \mathcal{N} is called the *Chaplygin reducing multiplier*.

Nonholonomic systems on unimodular Lie groups with right-invariant constraints and left-invariant metrics, so called *LR systems*, always have an invariant measure [30]. A nontrivial example of a nonholonomic LR system on the group $SO(n)$ (*n-dimensional Veselova problem*), which can be regarded also as a $SO(n-1)$ -Chaplygin system such that the reduced system on $S^{n-1} = SO(n)/SO(n-1)$ is Hamiltonian after a time rescaling, is given in [16] (see also Section 5).

The Chaplygin-type reduction and a (partial) Hamiltonization can be performed also for a class of K -invariant nonholonomic systems $(Q, \kappa, V, \mathcal{D})$, where the condition (5) is not satisfied on some K -invariant subvariety $S \subset Q$ (see [17]).

1.3. Chaplygin Sphere and Reduction of Internal Symmetries. The n -dimensional Chaplygin sphere describes the rolling without slipping of an n -dimensional balanced ball on an $(n-1)$ -dimensional hyperspace \mathcal{H} in \mathbb{R}^n ([14], see Section 2 below). This is an \mathbb{R}^{n-1} -Chaplygin system: the kinetic energy and the nonholonomic distribution \mathcal{D} are invariant with respect to the translations of the ball over the hyperplane \mathcal{H} . After \mathbb{R}^{n-1} -reduction it becomes the almost Hamiltonian system (6) on the cotangent bundle of the orthogonal group $SO(n)$,

$$(7) \quad \begin{array}{ccc} \mathbb{R}^{n-1} & \longrightarrow & \mathcal{D} \subset T(SO(n) \times \mathbb{R}^{n-1}) \\ & & \downarrow \\ & & \mathcal{D}/\mathbb{R}^{n-1} \cong TSO(n) \cong T^*SO(n). \end{array}$$

The system is additionally invariant with respect to the $SO(n-1)$ -action - rotations of the ball around the vertical vector Γ . The associated vector fields $\xi_{SO(n) \times \mathbb{R}^{n-1}}$ are sections of the connection (7) and we have Noether integrals (4) that descend to the conservation law $\dot{\Phi} = 0$ of the reduced flow. Here

$$(8) \quad \Phi : T^*SO(n) \rightarrow so(n-1)^*$$

is the equivariant momentum mapping of the $SO(n-1)$ -action with respect to the canonical form Ω on $T^*SO(n)$.

However, Φ is not the momentum mapping with respect to the nonholonomic form $\Omega + \Xi$. Recently, Hochgerner and Garcia-Naranjo proved that the form Ξ can be *truncated* to the form $\tilde{\Xi}$, such that Φ is the momentum mapping of the $SO(n-1)$ -action on $(T^*SO(n), \Omega + \tilde{\Xi})$ [22]. Moreover, the reduced system is almost Hamiltonian with respect to $\Omega + \tilde{\Xi}$ as well: $i_{X_{red}}(\Omega + \tilde{\Xi}) = dH_{red}$.

As a result, following the lines of the usual symplectic reduction, we can use the momentum mapping Φ to reduce the system to the almost Hamiltonian system on $(M_\eta, \mathbf{w}_\eta)$, where $M_\eta = \Phi^{-1}(\eta)/SO(n-1)_\eta$, $SO(n-1)_\eta$ is the coadjoint isotropy group of $\eta \in so(n-1)^*$:

$$(9) \quad i_{X_{red}^\eta} \mathbf{w}_\eta = dH_{red}^\eta$$

(see [22]). Now H_{red}^η is the induced Hamiltonian function on M_η . So, the Chaplygin multiplier method is still applicable. In particular, if the ball is homogeneous, the reduced forms \mathbf{w}_η are closed and the reduced systems (9) are Hamiltonian without a time reparametrization.

Let \mathcal{O}_η be the coadjoint orbit of η . The reduced space M_η is a \mathcal{O}_η -bundle over $T^*S^{n-1} \cong T^*(SO(n)/SO(n-1))$ that can be seen as a submanifold of the $SO(n-1)$ -reduced space $so(n)^* \times S^{n-1}$:

$$(10) \quad \begin{array}{ccc} \mathcal{O}_\eta & \longrightarrow & M_\eta \cong \Phi^{-1}(\mathcal{O}_\eta)/SO(n-1) & \subset & (T^*SO(n))/SO(n-1) \\ & & \downarrow & & \parallel \\ & & T^*S & & so(n)^* \times S^{n-1} \end{array} .$$

We shall consider the simplest but still very interesting and nontrivial case, when $\eta = 0$. Then the manifold M_0 is diffeomorphic to the cotangent bundle of the sphere S^{n-1} and the reduced form \mathbf{w}_0 is a semi-basic perturbation of the canonical symplectic form ω of T^*S^{n-1} .

For the sake of simplicity, denote \mathbf{w}_0 , H_{red}^0 , X_{red}^0 , by \mathbf{w} , H , X , respectively.

1.4. Outline and Results of the Paper. In Section 2, we recall the equations of motion of the Chaplygin sphere. The reduction of the system to the cotangent bundle of the sphere T^*S^{n-1} , for a zero value of the $SO(n-1)$ -momentum mapping Φ is described in Section 3.

The calculation of an invariant measure as well as the time reparametrization $d\tau = \mathcal{N}dt$ and the reduction of the system to the Hamiltonian form for a specific choice of an inertia operator I of the ball is given in Section 4. On the level of forms, this means that the form \mathbf{w} is conformally symplectic: $d(\mathcal{N}\mathbf{w}) = 0$. The description of the Hamiltonization is given in redundant variables, by the use of a Dirac bracket.

We show that the obtained Hamiltonian system is an integrable geodesic flow. Moreover, as in the 3-dimensional case [12], the reduced system is closely related to the *associated* nonholonomic Veselova problem (see Section 5). Namely, the reduced Veselova problem and the reduced Chaplygin sphere problem share the same toric foliation of T^*S^{n-1} .

In the 3-dimensional case, the group $SO(2)$ is Abelian and all reduced spaces M_η are diffeomorphic to T^*S^2 . After a remarkable change of variables, Chaplygin transformed the problem to the case $\eta = 0$ [8]. Since for $n > 3$ and $\eta \neq 0$ the coadjoint orbits \mathcal{O}_η are nontrivial, some additional efforts are needed for understanding the complete dynamics of the ball and it rest still unsolved.

2. CHAPLYGIN SPHERE

2.1. Kinematics. Following [14, 18], consider the Chaplygin-sphere problem of rolling without slipping of an n -dimensional balanced ball (the mass center C coincides with the geometrical center) of radius ρ on an $(n-1)$ -dimensional hyperspace \mathcal{H} in \mathbb{R}^n . For the configuration space we take the direct product of Lie groups $SO(n)$ and \mathbb{R}^n , where $g \in SO(n)$ is the rotation matrix of the sphere (mapping a frame attached to the body to the space frame) and $r \in \mathbb{R}^n$ is the position vector of its center C (in the space frame). For a trajectory $(g(t), r(t))$ define angular velocities of the sphere in the moving and the fixed frame, and the velocity in the fixed frame by

$$\omega = g^{-1}\dot{g}, \quad \Omega = \dot{g}g^{-1}, \quad \mathbf{V} = \dot{r}.$$

In what follows we identify $so(n) \cong so(n)^*$ by an invariant scalar product

$$(11) \quad \langle X, Y \rangle = -\frac{1}{2}\text{tr}(XY).$$

Let $I : so(n) \rightarrow so(n)^* \cong so(n)$ be the inertia tensor and m mass of the ball. The Lagrangian of the system is then given by

$$(12) \quad L = \frac{1}{2}\kappa((\dot{\omega}, \dot{\mathbf{V}}), (\dot{\omega}, \dot{\mathbf{V}})) = \frac{1}{2}\langle I\omega, \omega \rangle + \frac{1}{2}m(\mathbf{V}, \mathbf{V}),$$

where (\cdot, \cdot) is the Euclidean scalar product in \mathbb{R}^n .

Let $\Gamma \in \mathbb{R}^n$ be a *vertical* unit vector (considered in the fixed frame) orthogonal to the hyperplane \mathcal{H} and directed from \mathcal{H} to the center C . The condition for the sphere to roll without slipping leads that the velocity of the contact point is equal to zero:

$$(13) \quad \mathbf{V} - \rho\Omega\Gamma = 0.$$

The distribution

$$\mathcal{D} = \{(g, r, \omega, \mathbf{V}) \mid \mathbf{V} = \rho \text{Ad}_g(\omega)\Gamma\}$$

is right $(SO(n) \times \mathbb{R}^n)$ -invariant, so the Chaplygin sphere is an example of a coupled nonholonomic LR system on the direct product $SO(n) \times \mathbb{R}^n$ (see [25]).

If we take the fixed orthonormal base E_1, \dots, E_n such that $\Gamma = E_n$, then the constraint (13) takes the form $\dot{r}_i = \mathbf{V}_i = \rho\Omega_{in}$, $i = 1, \dots, n-1$, $\dot{r}_n = \mathbf{V}_n = 0$, where $\Omega_{ij} = \langle \Omega, E_i \wedge E_j \rangle$ ($X \wedge Y = X \otimes Y - Y \otimes X = XY^T - YX^T$, $X, Y \in \mathbb{R}^n$).

The last constraint is holonomic, and for the physical motion we take $r_n = \rho$.

From now on we take $SO(n) \times \mathbb{R}^{n-1}$ for the configuration space of the rolling sphere, where \mathbb{R}^{n-1} is identified with the affine hyperplane $\rho\Gamma + \mathcal{H}$. Then the

Chaplygin sphere is an \mathbb{R}^{n-1} -Chaplygin system (7), where the reduced Lagrangian reads

$$(14) \quad L_{red}(\omega, g) = \frac{1}{2}\langle I\omega, \omega \rangle + \frac{m\rho^2}{2}(\text{Ad}_g(\omega)\Gamma, \text{Ad}_g(\omega)\Gamma) =: \frac{1}{2}\langle \kappa_{red}(g)\omega, \omega \rangle.$$

Remark 1. We can also consider the *rubber Chaplygin sphere*, defined as a system (12), (13) subjected to the additional right-invariant constraints $\Omega_{ij} = 0$, $1 \leq i < j \leq n-1$ describing the no-twist condition at the contact point [11, 25].

2.2. Dynamics. From the constraints (13) we find the form of reaction forces in the right-trivialization in which the equations (2) become

$$(15) \quad \dot{M} = -\rho\Lambda \wedge \Gamma,$$

$$(16) \quad m\dot{\mathbf{V}} = \Lambda,$$

$$(17) \quad \dot{g} = \Omega \cdot g,$$

$$(18) \quad \dot{r} = \mathbf{V}.$$

where $M = \text{Ad}_g(I\omega) \in so(n)^* \cong so(n)$ is the ball angular momentum in the space and $\Lambda \in \mathbb{R}^n$ is the Lagrange multiplier.

Differentiating the constraints (13) and using (16) we get $\Lambda = m\rho\dot{\Omega}\Gamma$. On the other hand

$$(19) \quad \Lambda \wedge \Gamma = m\rho(\dot{\Omega}\Gamma) \wedge \Gamma = m\rho\left(\dot{\Omega}\Gamma \otimes \Gamma + \Gamma \otimes \Gamma\dot{\Omega}\right) = m\rho \text{pr}_{\mathfrak{h}}(\dot{\Omega}),$$

where $\mathfrak{h} \subset so(n)$ is the linear subspace $\mathfrak{h} = \mathbb{R}^n \wedge \Gamma$ and $\text{pr}_{\mathfrak{h}} : so(n) \rightarrow \mathfrak{h}$, $\text{pr}_{\mathfrak{h}}(\xi) = (\xi\Gamma) \wedge \Gamma = \xi\Gamma \otimes \Gamma + \Gamma \otimes \xi\Gamma$ is the orthogonal projection with respect to the scalar product (11).

Whence, (15), (17) is a closed system on $TSO(n)$, representing the Chaplygin reduction of the \mathbb{R}^{n-1} -symmetry. Now we need to write it in the left trivialization of $TSO(n)$.

Let $\gamma = g^{-1}\Gamma$ be the vertical vector in the frame attached to the ball. Then

$$(20) \quad \text{Ad}_{g^{-1}}(\mathfrak{h}) = \mathbb{R}^n \wedge \gamma =: \mathfrak{h}^\gamma.$$

From the identity

$$(21) \quad \dot{\omega} = \text{Ad}_{g^{-1}}(\dot{\Omega})$$

and the relations (19) and $\text{pr}_{\mathfrak{h}^\gamma}(\xi) = (\xi \cdot \gamma) \wedge \gamma = \xi\gamma \otimes \gamma + \gamma \otimes \xi\gamma$ we get

$$I\dot{\omega} = [I\omega, \omega] - m\rho^2(\dot{\omega}\gamma \otimes \gamma + \gamma \otimes \gamma\dot{\omega}).$$

Let us denote $m\rho^2$ by D and let

$$(22) \quad \mathbf{k} = I\omega + D\text{pr}_{\mathfrak{h}^\gamma}\omega = I\omega + D(\omega\gamma \otimes \gamma + \gamma \otimes \gamma\omega) \in so(n)^* \cong so(n)$$

be the angular momentum of the ball relative to the contact point (see [14]). Note that $\mathbf{k} = \kappa_{red}(g)\omega$, where the reduced metric $\kappa_{red}(g)$ is defined by (14).

By using the Poisson equation

$$(23) \quad \dot{\gamma} = -\omega\gamma$$

it easily follows $\frac{d}{dt}(\omega\gamma \otimes \gamma + \gamma \otimes \gamma\omega) = \dot{\omega}\gamma \otimes \gamma + \gamma \otimes \gamma\dot{\omega} + [\omega\gamma \otimes \gamma + \gamma \otimes \gamma\omega, \omega]$.

Therefore, the reduced Chaplygin sphere equations, in variables (\mathbf{k}, g) of the cotangent bundle $T^*SO(n)$ (or in variables (ω, g) of the tangent bundle $TSO(n)$) are given by

$$(24) \quad \dot{\mathbf{k}} = [\mathbf{k}, \omega],$$

$$(25) \quad \dot{g} = g \cdot \omega,$$

while the reduced kinetic energy is $H_{red}(\mathbf{k}, g) = \frac{1}{2}\langle \kappa_{red}^{-1}(g) \mathbf{k}, \mathbf{k} \rangle = \frac{1}{2}\langle \mathbf{k}, \omega(\mathbf{k}) \rangle$.

Let Ω be the canonical symplectic structure on $T^*SO(n)$, $d = \dim SO(n)$. It follows from [14, 15] that the reduced flow on $T^*SO(n)$ has an invariant measure

$$(26) \quad \varrho(\gamma)|_{\gamma=g^{-1}\Gamma} \Omega^d = 1/\sqrt{\det(\kappa_{red}(g))} \Omega^d = 1/\sqrt{\det(I + D\text{pr}_{\mathfrak{h}\gamma})}|_{\gamma=g^{-1}\Gamma} \Omega^d.$$

The system is additionally *left* $SO(n-1)$ -invariant where the action of $SO(n-1)$ is given by the rotations around the vertical vector Γ . The closed system (23), (24) in coordinates (\mathbf{k}, γ) represents the reduction of $SO(n-1)$ -symmetry to

$$(27) \quad so(n)^* \times S^{n-1} \cong (T^*SO(n))/SO(n-1).$$

The volume form (26) descends to the invariant measure

$$(28) \quad \varrho(\gamma) \Omega_{so(n)^*} \wedge \Omega_{S^{n-1}} = 1/\sqrt{\det(I + D\text{pr}_{\mathfrak{h}\gamma})} \Omega_{so(n)^*} \wedge \Omega_{S^{n-1}},$$

where $\Omega_{so(n)^*}$ and $\Omega_{S^{n-1}}$ are standard volume forms on $so(n)^*(\mathbf{k})$ and $S^{n-1}(\gamma)$, respectively (see [14, 15]).

2.3. Classical Chaplygin Sphere. In the case $n = 3$, under the isomorphism between \mathbb{R}^3 and $so(3)$

$$(29) \quad \vec{X} = (X_1, X_2, X_3) \longmapsto X = \begin{pmatrix} 0 & -X_3 & X_2 \\ X_3 & 0 & -X_1 \\ -X_2 & X_1 & 0 \end{pmatrix},$$

from (24) and (23) we obtain the classical Chaplygin's ball equations

$$(30) \quad \frac{d}{dt} \vec{\mathbf{k}} = \vec{\mathbf{k}} \times \vec{\omega}, \quad \frac{d}{dt} \vec{\gamma} = \vec{\gamma} \times \vec{\omega},$$

where $\vec{\mathbf{k}} = I\vec{\omega} + D\vec{\omega} - D(\vec{\omega}, \vec{\gamma})\vec{\gamma}$ and I is the inertia operator of the ball. In the space $(\vec{\mathbf{k}}, \vec{\gamma})$ the density of an invariant measure (28) is equal to

$$(31) \quad \varrho(\vec{\gamma}) = 1/\sqrt{\det(I + D\mathbf{I})(1 - D(\vec{\gamma}, (I + D\mathbf{I})^{-1}\vec{\gamma}))},$$

the expression given by Chaplygin in [8]. Since the system (30) has four integrals

$$(32) \quad F_1 = (\vec{\mathbf{k}}, \vec{\gamma}), \quad F_2 = (\vec{\gamma}, \vec{\gamma}) = 1, \quad F_3 = \frac{1}{2}(\vec{\mathbf{k}}, \vec{\omega}), \quad F_4 = (\vec{\mathbf{k}}, \vec{\mathbf{k}}),$$

it is integrable by the Euler-Jacobi theorem: the phase space \mathbb{R}^6 is almost everywhere foliated by invariant tori with quasi-periodic, non-uniform motion [1]. The integration in [8] is divided into the two steps. Firstly, equations (30) are solved in the case the area integral F_1 is zero, using elliptic coordinates on the Poisson sphere

$F_2 = 1$. Then, after an ingenious linear change of variables $(\vec{\mathbf{k}}, \vec{\gamma}) \mapsto (\vec{\mathbf{k}}_1, \vec{\gamma}_1)$, the problem transforms to the zero area case.

3. REDUCED SYSTEM IN REDUNDANT COORDINATES

3.1. **Reduction to T^*S^{n-1} .** From (15) we have

$$\begin{aligned} \frac{d}{dt}(\text{pr}_{so(n-1)} M) &= \frac{d}{dt}(\text{pr}_{so(n-1)} \text{Ad}_g(I\omega)) \\ (33) \quad &= \frac{d}{dt}(\text{pr}_{so(n-1)} \text{Ad}_g(I\omega + D \text{pr}_{\mathfrak{h}^\gamma} \omega)) \\ &= \frac{d}{dt}(\text{pr}_{so(n-1)} \text{Ad}_g \mathbf{k}) = 0, \end{aligned}$$

where $so(n-1) \subset so(n)$ is orthogonal complement to $\mathfrak{h} = \mathbb{R}^n \wedge \Gamma$ with respect to the scalar product (11).

The integral (33) is actually the momentum mapping (8) of the left $SO(n-1)$ -action. For $n = 3$ we have the classical area integral $F_1 = (\vec{\mathbf{k}}, \vec{\gamma}) = (I\vec{\omega}, \vec{\gamma})$.

So we can pass to the reduced system (9) on $M_\eta = \Phi^{-1}(\eta)/SO(n-1)_\eta$ ([22], see Introduction). We shall consider the simplest but still very interesting case, when we fix the value of the momentum mapping Φ to be zero

$$(34) \quad \text{pr}_{so(n-1)} \text{Ad}_g(I\omega) = \text{pr}_{so(n-1)} \text{Ad}_g \mathbf{k} = \text{pr}_{so(n-1)^\gamma} (I\omega) = \text{pr}_{so(n-1)^\gamma} \mathbf{k} = 0.$$

Here $so(n-1)^\gamma := \text{Ad}_{g^{-1}} so(n-1) = (\mathbb{R}^n \wedge \gamma)^\perp = (\mathfrak{h}^\gamma)^\perp$.

Whence, both \mathbf{k} and $I\omega$ belong to the subspace (20). Now, let us introduce new variables $p, \xi \in \mathbb{R}^n$ orthogonal to γ

$$(35) \quad (\gamma, p) = (\gamma, \xi) = 0,$$

such that

$$(36) \quad \mathbf{k} = \gamma \wedge p, \quad \omega = I^{-1}(\gamma \wedge \xi).$$

Lemma 1. *The variables p and ξ are related via*

$$(37) \quad p = \xi - DI^{-1}(\gamma \wedge \xi)\gamma$$

Proof. The proof directly follows from the definition $\mathbf{k} = I\omega + D((\omega\gamma) \wedge \gamma)$ and relations (36). \square

From (37), under the conditions (35), the variable ξ can be uniquely expressed via p and γ .

Note that the coordinates (γ, p) can be considered as redundant coordinates of the cotangent bundle of the sphere T^*S^{n-1} realized as a subvariety of \mathbb{R}^{2n} defined by constraints

$$(38) \quad \phi_1 \equiv (\gamma, \gamma) = 1, \quad \phi_2 \equiv (\gamma, p) = 0.$$

Theorem 2. *The reduced Chaplygin-sphere problem on $T^*S^{n-1} = \Phi^{-1}(0)/SO(n-1)$ is described by the equations*

$$(39) \quad \dot{\gamma} = X_\gamma(\gamma, p) = -\omega\gamma = -I^{-1}(\gamma \wedge \xi(\gamma, p))\gamma$$

$$(40) \quad \dot{p} = X_p(\gamma, p) = -\omega p = -I^{-1}(\gamma \wedge \xi(\gamma, p))p$$

Proof. The mapping $(\gamma, p) \mapsto (\mathbf{k} = \gamma \wedge p, \gamma)$ realizes T^*S^{n-1} as a submanifold of (27) (see diagram (10)). The equation (39) follows directly from the Poisson equation (23). On the other hand, from the equation (24) we get

$$\begin{aligned} \dot{\gamma} \wedge p + \gamma \wedge \dot{p} &= [\gamma \wedge p, \omega] \\ \implies -\omega\gamma p^T - p(-\omega\gamma)^T + \gamma \wedge \dot{p} &= \gamma p^T \omega - p\gamma^T \omega - \omega\gamma p^T + \omega p\gamma^T \\ \implies \gamma \wedge \dot{p} &= \omega p\gamma^T + \gamma p^T \omega = (\omega p) \wedge \gamma \\ \implies \dot{p} &= -\omega p + \lambda\gamma. \end{aligned}$$

The multiplier λ is equal to zero. Indeed, from (38) we have

$$\frac{d}{dt}\phi_2 = (\dot{\gamma}, p) + (\gamma, \dot{p}) = (-\omega\gamma, p) + (\gamma, -\omega p) + \lambda(\gamma, \gamma) = \lambda = 0.$$

□

Note that the reduced Hamiltonian

$$(41) \quad H(\gamma, p) = \frac{1}{2}\langle \mathbf{k}, \omega \rangle = \frac{1}{2}\langle \gamma \wedge p, I^{-1}(\gamma \wedge \xi(\gamma, p)) \rangle$$

(which is now unique only on the subvariety (38)) as well as the system (39), (40) itself, is defined on

$$(42) \quad \hat{\mathbb{R}}^{2n} = \mathbb{R}^{2n} \setminus \{\gamma = 0\}.$$

Also considered on $\hat{\mathbb{R}}^{2n}$, the *extended system* (39), (40) preserves the functions ϕ_1, ϕ_2 , the Hamiltonian (41) and the reduced momentum

$$(43) \quad K(\gamma, p) = \langle \gamma \wedge p, \gamma \wedge p \rangle = (\gamma, \gamma)(p, p) - (\gamma, p)^2.$$

3.2. Chaplygin Reducing Multiplier. At the points of T^*S^{n-1} , the vector field $X = (X_\gamma, X_p)$ of the system (39), (40) can be written in the almost Hamiltonian form $i_X(\mathbf{w}) = dH$, where the form \mathbf{w} is a non-degenerate 2-form on T^*S^{n-1} , a semi-basic perturbation of the canonical symplectic form

$$(44) \quad \omega = dp_1 \wedge d\gamma_1 + \cdots + dp_n \wedge d\gamma_n |_{T^*S^{n-1}}$$

(see [22]).

Let \mathbf{w} be an almost symplectic form, i.e., a nondegenerate 2-form on an even dimensional manifold M . For an almost Hamiltonian flow $\dot{x} = X$, $i_X\mathbf{w} = dH$, the *Chaplygin multiplier* is a nonvanishing function \mathcal{N} such that $\tilde{\omega} = \mathcal{N}\mathbf{w}$ is closed. Since $i_{\tilde{X}}\tilde{\omega} = dH$, $\tilde{X} = \frac{1}{\mathcal{N}}X$, applying the time substitution $d\tau = \mathcal{N}dt$, the system $\dot{x} = X$ becomes the Hamiltonian system $\frac{d}{d\tau}x = \tilde{X}$ with respect to the symplectic form $\tilde{\omega}$ [28, 7, 29, 11]. More generally, \mathcal{N} is the Chaplygin multiplier if there exist a 2-form $\hat{\mathbf{w}}$ such that $i_X\hat{\mathbf{w}} = 0$ and $\tilde{\omega} = \mathcal{N}(\mathbf{w} - \hat{\mathbf{w}})$ is symplectic (see [11]). Then, as

above, the system $\dot{x} = X$ becomes the Hamiltonian system $\frac{d}{d\tau}x = \tilde{X}$ with respect to the symplectic form $\tilde{\omega}$.

Alternatively, a transparent and classical way to introduce the Chaplygin reducing multiplier for our system is as follows (e.g., see Section 3 in [16]). Let $\mathcal{N}(\gamma)$ be a differentiable nonvanishing positive function in a neighborhood of S^{n-1} . Consider the coordinate transformation

$$(\gamma, p) \longmapsto (\gamma, \tilde{p}), \quad \tilde{p} = \mathcal{N}p$$

defined in some neighborhood of T^*S^{n-1} and the new symplectic form

$$\begin{aligned} (45) \quad \tilde{\omega} &= d\tilde{p}_1 \wedge d\gamma_1 + \cdots + d\tilde{p}_n \wedge d\gamma_n|_{T^*S^{n-1}} \\ &= \mathcal{N}\omega + p_1 d\mathcal{N} \wedge d\gamma_1 + \cdots + p_n d\mathcal{N} \wedge d\gamma^n|_{T^*S^{n-1}}. \end{aligned}$$

Then \mathcal{N} is a *Chaplygin multiplier* for the reduced system if the equations (39), (40) in the new time $d\tau = \mathcal{N}(q)dt$ becomes Hamiltonian with respect to the form $\tilde{\omega}$. If \mathcal{N} is a Chaplygin multiplier then from the Liouville theorem we have

$$(46) \quad \mathcal{L}_{\tilde{X}}(\tilde{\omega}^{n-1}) = 0 \iff \mathcal{L}_X(\mathcal{N}^{n-2}\omega^{n-1}) = 0,$$

i.e., the original system has the invariant measure with density $\mathcal{N}(\gamma)^{n-2}$. Further, the form \mathbf{w} reads

$$(47) \quad \mathbf{w} = \omega + p_1 d\ln\mathcal{N} \wedge d\gamma_1 + \cdots + p_n d\ln\mathcal{N} \wedge d\gamma^n|_{T^*S^{n-1}}.$$

3.3. Homogeneous Sphere. It is proved in [22] that the reduced equations of motion (9) of the homogeneous ball are already Hamiltonian, for any value of the $SO(n-1)$ -momentum mapping. This interesting result, for $\Phi = \eta = 0$ can be easily derived from Theorem 2.

Suppose the inertia operator I equals $s\mathbf{I}$ (multiplication by a constant $s > 0$). Then the equation (37), under the conditions (35), gives $\xi = sp/(s+D)$. The reduced system (39), (40) takes the form

$$(48) \quad \dot{\gamma} = \frac{1}{s+D}p, \quad \dot{p} = -\frac{(p, p)}{s+D}\gamma,$$

representing the geodesic flow of the standard $SO(n)$ -invariant metric of the sphere multiplied by $s+D$. Note that in this case the angular velocity

$$\omega = \frac{1}{s}(\gamma \wedge \xi) = \frac{1}{s+D}(\gamma \wedge p)$$

is constant along the flow of (48). Actually, the angular velocity ω is constant for the rolling of the homogeneous ball for any value of $SO(n-1)$ -momentum mapping. Namely, substituting $M = s\Omega$ into the equations (15) and (19) we obtain

$$\text{pr}_{\mathfrak{h}}(s\dot{\Omega} + D\dot{\Omega}) = 0, \quad \text{pr}_{\mathfrak{h}^\perp}(s\dot{\Omega}) = 0,$$

which implies $\dot{\Omega} = 0$. Further, from (16), (19), (21) we get $\dot{\omega} = \dot{\mathbf{V}} = 0$ (see also [22]).

4. HAMILTONIZATION

In this section we shall perform the Hamiltonization of the reduced Chaplygin sphere (39), (40) for the inertia operator defined on the base $E_i \wedge E_j$ via

$$(49) \quad I(E_i \wedge E_j) = \frac{a_i a_j D}{D - a_i a_j} E_i \wedge E_j, \quad 1 \leq i < j \leq n,$$

where $0 < a_i a_j < D$, $1 \leq i, j \leq n$.

The form of the inertia operator as well as the form of the Chaplygin multiplier below is motivated by the corresponding formulas in the problem of motion of the n -dimensional Veselova problem as well as the rubber Chaplygin ball given in [16] and [25], respectively.

Let $A = \text{diag}(a_1, \dots, a_n)$.

In the tree-dimensional case the operator (49) defines a generic rigid body inertia tensor I . Indeed, using the isomorphism (29), we get

$$(50) \quad I = \text{diag}(I_1, I_2, I_3)$$

where $I_1 = a_2 a_3 D / (D - a_2 a_3)$, $I_2 = a_3 a_1 D / (D - a_3 a_1)$, $I_3 = a_2 a_3 D / (D - a_2 a_3)$.

Conversely, given a generic inertia tensor (50) (one can always assume that the axes of the frame attached to the ball are principal axes of inertia), the matrix $A = \text{diag}(a_1, a_2, a_3)$ is determined via

$$(51) \quad a_i = \sqrt{I_1 I_2 I_3 D} (I_i + D) / I_i \sqrt{(I_1 + D)(I_2 + D)(I_3 + D)}, \quad i = 1, 2, 3.$$

Remark 2. In general, for $n \geq 4$, the operator (49) is not a physical inertia operator of a multidimensional rigid body (see [14]). However, by taking conditions

$$(52) \quad a_1 = a_2 = \dots = a_{n-1} \neq a_n.$$

and $2a_n D > a_1 a_n + a_1 D$, we get the operator $I\omega = J\omega + \omega J$, where

$$J = \text{diag}(J_1, \dots, J_1, J_n), \quad J_1 = \frac{a_1^2 D}{2(D - a_1^2)}, \quad J_n = \frac{a_1 a_n D}{D - a_1 a_n} - \frac{a_1^2 D}{2(D - a_1^2)},$$

representing a $SO(n-1)$ -symmetric rigid body (*multidimensional Lagrange case* [2]) with a *mass tensor* J .

Theorem 3. *The extended reduced Chaplygin sphere equations (39), (40), defined by the inertia tensor (49), read*

$$(53) \quad \dot{\gamma} = \frac{1}{D} p - \frac{(p, \gamma)}{D(\gamma, A^{-1}\gamma)} A^{-1}\gamma + \frac{(\gamma, Ap)}{D^2(\gamma, A^{-1}\gamma)} \gamma - \frac{(\gamma, \gamma)}{D^2(\gamma, A^{-1}\gamma)} Ap,$$

$$(54) \quad \dot{p} = \frac{(p, A^{-1}\gamma)}{D(\gamma, A^{-1}\gamma)} p - \frac{(p, p)}{D(\gamma, A^{-1}\gamma)} A^{-1}\gamma + \frac{(p, Ap)}{D^2(\gamma, A^{-1}\gamma)} \gamma - \frac{(p, \gamma)}{D^2(\gamma, A^{-1}\gamma)} Ap.$$

Proof. From the definition (49), the angular velocity is given by

$$(55) \quad \omega = I^{-1}(\gamma \wedge \xi) = A^{-1}\gamma \wedge A^{-1}\xi - \frac{1}{D}\gamma \wedge \xi.$$

Now, the equation (37), under the conditions (35), can be solved

$$(56) \quad \xi = \frac{1}{D(\gamma, A^{-1}\gamma)} (Ap - (p, A\gamma)\gamma).$$

Thus $\omega = (A^{-1}\gamma \wedge p - \gamma \wedge Ap/D) / D(\gamma, A^{-1}\gamma)$ and (53), (54) simply follows from (39), (40). \square

4.1. Invariant Measure. The canonical volume form Ω on \mathbb{R}^{2n} induces the volume form σ on $T^*S^{n-1} \subset \mathbb{R}^{2n}$ (e.g., see paragraph 3.6, Ch. 1 [1]). A simple description of σ , in terms of the restricted symplectic structure (44) is as follows.

Consider the standard spherical coordinates $(\theta, r) = (\theta_1, \dots, \theta_{n-1}, r)$ on $\mathbb{R}^n(\gamma)$ and the corresponding canonical momenta $(\pi_\theta, \pi_r) = (\pi_1, \dots, \pi_{n-1}, \pi_r)$ on $\mathbb{R}^{2n}(\gamma, p)$ with respect to the canonical symplectic form:

$$dp_1 \wedge d\gamma_1 + \dots + dp_n \wedge d\gamma_n = d\pi_1 \wedge d\theta_1 + \dots + d\pi_{n-1} \wedge d\theta_{n-1} + d\pi_r \wedge dr.$$

Then the volume form Ω can be represented as

$$(57) \quad \Omega = dp_1 \wedge d\gamma_1 \wedge \dots \wedge dp_n \wedge d\gamma_n = (d\pi_1 \wedge d\theta_1 \wedge \dots \wedge d\pi_{n-1} \wedge d\theta_{n-1}) \wedge d\pi_r \wedge dr,$$

where $r = \sqrt{(\gamma, \gamma)}$ and $p_r = (\gamma, p) / \sqrt{(\gamma, \gamma)}$. The coordinates (θ, π_θ) are canonical coordinates (the symplectic form (44) equals $d\pi_1 \wedge d\theta_1 + \dots + d\pi_{n-1} \wedge d\theta_{n-1}$) and

$$\sigma = d\pi_1 \wedge d\theta_1 \wedge \dots \wedge d\pi_{n-1} \wedge d\theta_{n-1}$$

is the canonical volume form on the cotangent bundle T^*S^{n-1} , naturally extended to $\hat{\mathbb{R}}^{2n}$.

Proposition 4. *The reduced Chaplygin system (53), (54) on T^*S^{n-1} possesses an invariant measure*

$$(58) \quad \mu(\gamma) \sigma = (A^{-1}\gamma, \gamma)^{-(n-2)/2} \sigma.$$

Proof. The divergence of the vector field X in $\hat{\mathbb{R}}^{2n}$ is

$$(59) \quad \text{div}(X) = \sum_{i=1}^n \left(\frac{\partial \dot{\gamma}_i}{\partial \gamma_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right) = (n-2) \left(\frac{(\gamma, A^{-1}p)}{D(\gamma, A^{-1}\gamma)} + \frac{(\gamma, Ap)}{D^2(\gamma, A^{-1}\gamma)} \right) + \Psi,$$

where

$$\Psi = \left(\frac{2(A^{-2}\gamma, \gamma)}{D(\gamma, A^{-1}\gamma)^2} + \frac{2(\gamma, \gamma)}{D^2(\gamma, A^{-1}\gamma)^2} - \frac{\text{tr}A^{-1}}{D(\gamma, A^{-1}\gamma)} - \frac{\text{tr}A}{D^2(\gamma, A^{-1}\gamma)} \right) (\gamma, p).$$

Whence, on the invariant submanifold $\phi_2 = \pi_r = 0$, in view of (53), we get

$$\sum_{i=1}^n \left(\frac{\partial \dot{\gamma}_i}{\partial \gamma_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right) = (n-2) \frac{(\gamma, A^{-1}\dot{\gamma})}{(\gamma, A^{-1}\gamma)} = -\frac{\dot{\mu}}{\mu}.$$

In other words, the density $\mu(\gamma)$ satisfies the Liouville equation

$$(60) \quad \text{div}(\mu X) = \sum_{i=1}^n \dot{q}_i \frac{\partial \mu}{\partial q_i} + \mu \sum_{i=1}^n \left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right) = 0$$

on the manifold $\phi_2 = \pi_r = 0$.

On the other side, from (57) we obtain

$$(61) \quad \mathcal{L}_X(\mu\Omega) = \mathcal{L}_X(\mu\sigma) \wedge d\pi_r \wedge dr + \mu\sigma \wedge \mathcal{L}_X(d\pi_r \wedge dr).$$

Since the functions ϕ_1, ϕ_2 are invariants of the vector field X , the Lie derivatives $\mathcal{L}_X d\pi_r$ and $\mathcal{L}_X dr$ equal zero. Further, (60) implies that the left hand side of (61) is also equal to zero on the invariant subvariety $\phi_2 = \pi_r = 0$. Thus we conclude

$$\mathcal{L}_X(\mu\sigma)|_{T^*S^{n-1}} = 0$$

as required. \square

Remark 3. The reduced vector field (9) has an invariant measure for $\Phi = \eta \neq 0$ as well. Namely, the $SO(n-1)$ -reduced system (23), (24) preserve the volume form (28). Then the restriction of the flow to the invariant manifold M_η (see diagram (10)) preserves the *induced volume form* (e.g., see [1]). In this sense, Proposition 4 is equivalent to the proportionality of the densities of measures (28) and (58) (compare with Theorem 5.1 [16]). In particular, for $n = 3$ the density $\mu(\gamma)$ after the transformation (51), up to a multiplication by a constant, takes the form (31).

4.2. Time Reparametrization. The reduced Hamiltonian (41) read

$$(62) \quad H(\gamma, p) = \frac{1}{2D(\gamma, A^{-1}\gamma)} \langle \gamma \wedge p, A^{-1}\gamma \wedge p - \frac{1}{D}\gamma \wedge Ap \rangle.$$

According to the constraints (38), instead of (62) we can use the Hamiltonian function

$$(63) \quad H(\gamma, p) = \frac{1}{2D^2(\gamma, A^{-1}\gamma)} (D(\gamma, A^{-1}\gamma)(p, p) - (p, Ap))$$

As follows from Proposition 4 and the relation (46), if the reduced Chaplygin system on T^*S^{n-1} is transformable to a Hamiltonian form by a time reparameterization, then the corresponding reducing multiplier \mathcal{N} should be proportional to $1/\sqrt{(\gamma, A^{-1}\gamma)}$.

Theorem 5. *Under the time substitution*

$$(64) \quad d\tau = \mathcal{N} dt = \frac{1}{D\sqrt{(A^{-1}\gamma, \gamma)}} dt$$

and an appropriate change of momenta

$$(65) \quad (\gamma, p) \mapsto (\gamma, \tilde{p}), \quad \tilde{p} = \frac{1}{D\sqrt{(\gamma, A^{-1}\gamma)}} p$$

*the reduced system (53), (54) becomes a Hamiltonian system describing a geodesic flow on T^*S^{n-1} with the Hamiltonian*

$$(66) \quad H(\gamma, \tilde{p}) = \frac{1}{2} (D(\gamma, A^{-1}\gamma)(\tilde{p}, \tilde{p}) - (\tilde{p}, A\tilde{p})).$$

Proof. Consider the cotangent bundle T^*S^{n-1} realized as a submanifold of \mathbb{R}^{2n} given by

$$(67) \quad \psi_1 \equiv (\gamma, \gamma) = 1, \quad \psi_2 \equiv (\gamma, \tilde{p}) = 0.$$

The canonical Poisson bracket on T^*S^{n-1} with respect to the symplectic form (45) can be described by the use of the Dirac bracket (see [10, 26, 1]):

$$\{F, G\}_d = \{F, G\} - (\{F, \psi_1\}\{G, \psi_2\} - \{F, \psi_2\}\{G, \psi_1\})/\{\psi_1, \psi_2\},$$

where

$$\{F, G\} = \sum_{i=1}^n \left(\frac{\partial F}{\partial \gamma_i} \frac{\partial G}{\partial \tilde{p}_i} - \frac{\partial F}{\partial \tilde{p}_i} \frac{\partial G}{\partial \gamma_i} \right).$$

Considered on $\hat{\mathbb{R}}^{2n}$, the bracket $\{\cdot, \cdot\}_d$ is degenerate and has two Casimir functions ψ_1 and ψ_2 . The symplectic leaf given by (67) is exactly the cotangent bundle T^*S^{n-1} endowed with the canonical symplectic form.

Under the mapping (65), the Hamiltonian (63) transforms to (66). With the above notation, the geodesic flow defined by Hamiltonian function (66), in the time τ , is the restriction to (67) of

$$(68) \quad \gamma'_i = \frac{d}{d\tau} \gamma_i = \{\gamma_i, H\}_d, \quad \tilde{p}'_i = \frac{d}{d\tau} \tilde{p}_i = \{\tilde{p}_i, H\}_d, \quad i = 1, \dots, n.$$

It is convenient to find equations (68) using the Lagrange multipliers (see [26, 1]). Introduce

$$H^* = H - \lambda \psi_1 - \mu \psi_2.$$

The equations (68) are then given by

$$\begin{aligned} \gamma' &= \frac{\partial H^*}{\partial \tilde{p}} = \frac{\partial H}{\partial \tilde{p}} - \mu \gamma = D(A^{-1}\gamma, \gamma)\tilde{p} - A\tilde{p} - \mu\gamma, \\ \tilde{p}' &= -\frac{\partial H^*}{\partial \gamma} = -\frac{\partial H}{\partial \gamma} + 2\lambda\gamma + \mu\tilde{p} = -D(\tilde{p}, \tilde{p})A^{-1}\gamma + 2\lambda\gamma + \mu\tilde{p} \end{aligned}$$

where the multipliers λ and μ are determined from the condition that the constraint functions ψ_1 and ψ_2 are integrals of the motion.

Straightforward calculations yield

$$\lambda = \frac{(A\tilde{p}, \tilde{p})}{2(\gamma, \gamma)}, \quad \mu = \frac{D(\gamma, A^{-1}\gamma)(\tilde{p}, \gamma) - (A\tilde{p}, \gamma)}{(\gamma, \gamma)}$$

and therefore

$$(69) \quad \gamma' = D(A^{-1}\gamma, \gamma)\tilde{p} - A\tilde{p} + \frac{(\gamma, A\tilde{p})}{(\gamma, \gamma)}\gamma - \frac{D(\gamma, A^{-1}\gamma)(\tilde{p}, \gamma)}{(\gamma, \gamma)}\gamma,$$

$$(70) \quad \tilde{p}' = -D(\tilde{p}, \tilde{p})A^{-1}\gamma + \frac{(\tilde{p}, A\tilde{p})}{(\gamma, \gamma)}\gamma - \frac{(\gamma, A\tilde{p})}{(\gamma, \gamma)}\tilde{p} + \frac{D(\gamma, A^{-1}\gamma)(\tilde{p}, \gamma)}{(\gamma, \gamma)}\tilde{p}.$$

In the time t , inverting the mapping (65), the equation (69) takes the form

$$\begin{aligned} \dot{\gamma} \cdot D\sqrt{(\gamma, A^{-1}\gamma)} &= \frac{1}{D\sqrt{(\gamma, A^{-1}\gamma)}} \\ &\cdot \left(D(A^{-1}\gamma, \gamma)p - Ap + \frac{(\gamma, Ap)}{(\gamma, \gamma)}\gamma - \frac{D(\gamma, A^{-1}\gamma)(p, \gamma)}{(\gamma, \gamma)}\gamma \right), \end{aligned}$$

i.e.,

$$(71) \quad \dot{\gamma} = \frac{1}{D}p - \frac{1}{D^2(\gamma, A^{-1}\gamma)}Ap + \frac{(\gamma, Ap)}{D^2(\gamma, A^{-1}\gamma)(\gamma, \gamma)}\gamma - \frac{(p, \gamma)}{D(\gamma, \gamma)}\gamma,$$

which coincides with (53) at the points of T^*S^{n-1} . Further,

$$(72) \quad \begin{aligned} \frac{d}{d\tau}\tilde{p} &= \frac{d}{d\tau}\left(\frac{p}{D\sqrt{(\gamma, A^{-1}\gamma)}}\right) = \frac{d}{dt}\left(\frac{p}{D\sqrt{(\gamma, A^{-1}\gamma)}}\right)D\sqrt{(\gamma, A^{-1}\gamma)} \\ &= \left(p\frac{d}{dt}\frac{1}{\sqrt{(\gamma, A^{-1}\gamma)}} + \dot{p}\frac{1}{\sqrt{(\gamma, A^{-1}\gamma)}}\right)\sqrt{(\gamma, A^{-1}\gamma)} = \dot{p} - p\frac{(A^{-1}\gamma, \dot{\gamma})}{(\gamma, A^{-1}\gamma)}. \end{aligned}$$

Finally, substituting $\tilde{p} = \mathcal{N}p$ into the right hand side of (70), combining with (71) and (72), we get

$$(73) \quad \begin{aligned} \dot{p} &= -\frac{(p, p)}{D(\gamma, A^{-1}\gamma)}A^{-1}\gamma + \frac{(p, Ap)}{D^2(\gamma, A^{-1}\gamma)(\gamma, \gamma)}\gamma \\ &\quad + \frac{(p, A^{-1}\gamma)}{D(\gamma, A^{-1}\gamma)}p - \frac{(\gamma, p)}{D^2(\gamma, A^{-1}\gamma)^2}p. \end{aligned}$$

As above, the equations (54) and (73) are different, but they coincide on the invariant manifold $\phi_1 = \psi_1 = 1$, $\phi_2 = \psi_2 = 0$. The theorem is proved. \square

Remark 4. The link between the Dirac bracket and the Lagrange multiplier approach can be expressed via

$$\lambda = \frac{\{H, \psi_2\}}{\{\psi_1, \psi_2\}}, \quad \mu = -\frac{\{H, \psi_1\}}{\{\psi_1, \psi_2\}}.$$

Also, note that the reduced almost symplectic form (47) is given by:

$$\mathbf{w} = \sum_{i,j=1}^n dp_i \wedge d\gamma_j - \frac{p_i a_j^{-1} \gamma_j}{(\gamma, A^{-1}\gamma)} d\gamma_j \wedge d\gamma_i |_{T^*S^{n-1}}.$$

Remark 5. During the referee process of this paper, the paper [23] appeared, where the Abelian \mathfrak{v} -Chaplygin systems associated to Cartan decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of semi-simple Lie algebras are studied. They are defined on the direct product of a Lie group K ($\mathfrak{k} = \text{Lie}(K)$) endowed with a left-invariant metric with the vector space \mathfrak{v} ($\mathfrak{v} = [\Gamma, \mathfrak{k}] \subset \mathfrak{p}$) endowed with the metric induced from the Killing form. Here $\Gamma \in \mathfrak{p}$ is fixed. As an example, taking the Cartan decomposition $so(n, 1) = so(n) \oplus \mathbb{R}^n$ of the Lie algebra $so(n, 1)$ one gets the Chaplygin sphere problem (compare with the equations (73) in [25]). Besides \mathfrak{v} -reduction to T^*K , likewise the Chaplygin sphere problem, the system has an internal symmetry group $H \subset K$ (isotropy group of Γ) and admits the almost symplectic reduction with respect to the H -action. Hochgerner derived the equations on the parameters of the kinetic energy, such that the (zero momentum) reduced almost symplectic form is conformally symplectic. The operator (49) represents the solution of these equations within the class of diagonal operators on $so(n)$ with respect to the base $E_i \wedge E_j$ [23].

5. INTEGRABILITY

5.1. Classical Chaplygin Sphere and the Veselova Problem. The *Veselova problem* describes the motion of a rigid body about a fixed point subject to the

nonholonomic constraint

$$(74) \quad (\vec{\mathbf{w}}, \vec{\gamma}) = 0,$$

where $\vec{\mathbf{w}}$ is the vector of the angular velocity in the body frame and $\vec{\gamma}$ is a representation of a unit vector fixed in a space, relative to the body frame [30]. The equations of motion in the moving frame have the form

$$(75) \quad \frac{d}{dt} \mathcal{I} \vec{\mathbf{w}} = \mathcal{I} \vec{\mathbf{w}} \times \vec{\mathbf{w}} + \lambda \vec{\gamma}, \quad \frac{d}{dt} \vec{\gamma} = \vec{\gamma} \times \vec{\mathbf{w}},$$

where \mathcal{I} is the inertia tensor of the rigid body and λ is a Lagrange multiplier chosen such that $\vec{\mathbf{w}}(t)$ satisfies the constraint (74),

$$(76) \quad \lambda = -\frac{(\mathcal{I} \vec{\mathbf{w}} \times \vec{\mathbf{w}}, \mathcal{I}^{-1} \vec{\gamma})}{(\mathcal{I}^{-1} \vec{\gamma}, \vec{\gamma})}.$$

Here we suppose that all eigenvalues of \mathcal{I} are greater than 1.

Equations (75), (76) also define a dynamical system on the whole space $\mathbb{R}^6(\vec{\mathbf{w}}, \vec{\gamma})$, and the constraint function $f_1 = (\vec{\mathbf{w}}, \vec{\gamma})$ appears as its first integral. The system has an invariant measure with density $\sqrt{(\mathcal{I}^{-1} \vec{\gamma}, \vec{\gamma})}$. Following [12], by introducing $\vec{K} = \mathcal{I} \vec{\mathbf{w}} - (\mathcal{I}^0 \vec{\mathbf{w}}, \vec{\gamma}) \vec{\gamma}$, $\mathcal{I}^0 = \mathcal{I} - \mathbf{I}$ one can write system (75), (76) in the form

$$(77) \quad \frac{d}{dt} \vec{K} = \vec{K} \times \vec{\mathbf{w}}, \quad \frac{d}{dt} \vec{\gamma} = \vec{\gamma} \times \vec{\mathbf{w}}.$$

Apart from $f_1 = (\vec{\mathbf{w}}, \vec{\gamma}) = (\vec{K}, \vec{\gamma})$, it always has the geometric integral $f_2 = (\vec{\gamma}, \vec{\gamma}) = 1$ and two other independent integrals

$$(78) \quad f_3 = \frac{1}{2} (\vec{K}, \vec{\mathbf{w}}) - \frac{1}{2} (\vec{K}, \vec{\gamma}) (\mathcal{I}^0 \vec{\mathbf{w}}, \vec{\gamma}), \quad f_4 = (\vec{K}, \vec{K}).$$

On the constraint subvariety (74), these functions reduce to the energy integral $\frac{1}{2} (\mathcal{I} \vec{\mathbf{w}}, \vec{\mathbf{w}})$ and $(I \vec{\mathbf{w}}, I \vec{\mathbf{w}}) - (I \vec{\mathbf{w}}, \vec{\gamma})^2$ (see [30]).

By the Euler-Jacobi theorem [1], the above system is solvable by quadratures on the whole space \mathbb{R}^6 . For $f_1 = 0$ the system was integrated by Veselova (e.g., one can find the motion using the isomorphism with a celebrated Neumann system [30]). Next, as was shown in [12], the restriction of the extended Veselova system (75), (76) onto the level variety $f_1 = c_1$ ($c_1 \neq 0$) can be reduced to this system on the level $f_1 = 0$ by a linear change of variables $(\vec{K}, \vec{\gamma}) \mapsto (\vec{K}_1, \vec{\gamma}_1)$ and an appropriate time reparametrization. This linear change was found by using a relation of the Veselova system with the Chaplygin sphere problem, which we are going to describe now.

Define the operator I and vector $\vec{\omega}$ by:

$$(79) \quad \mathcal{I} = \mathbf{I} + D I^{-1}, \quad \vec{\mathbf{w}} = -I \vec{\omega} \iff I = D(\mathcal{I} - \mathbf{I})^{-1}, \quad \vec{\omega} = -\frac{1}{D} (\mathcal{I} - \mathbf{I}) \vec{\mathbf{w}}.$$

Now we can state the following remarkable correspondence:

Theorem 6. (Fedorov [12]) *The invariant tori $f_1 = c_1, f_2 = 1, f_3 = c_3, f_4 = c_4$ of the Veselova problem (75), (76), via (79) transform to the invariant tori $F_1 = c_1, F_2 = 1, F_3 = c_3, F_4 = c_4$ of the Chaplygin sphere problem (30).*

Let us mention that there are two interesting isomorphisms between the Chaplygin sphere problem (30) with $F_1 = (\vec{\mathbf{k}}, \vec{\gamma}) = 0$ and the Clebsch case of the Kirchoffs equations of a rigid body motion in an ideal fluid, with a zero area integral. The first one is described in [13] and the other one is given recently in [5].

5.2. Veselova Problem on $SO(n)$. It appears that the analogue of Theorem 6 can be formulated for an arbitrary dimension n and a zero value of the momentum (34). First, for a reader's sake, we shall briefly recall some definitions and results of [16].

Consider a nonholonomic LR system on $SO(n)$ defined by the left-invariant Lagrangian $L_{\mathcal{I}}(g, \dot{g}) = \frac{1}{2}\langle \mathcal{I}\mathbf{w}, \mathbf{w} \rangle = -\frac{1}{4}\text{tr}(\mathcal{I}\mathbf{w}\mathbf{w})$ where $\mathcal{I} : so(n) \rightarrow so(n)$ is positive definite and the right-invariant distribution \mathcal{D}_r on $TSO(n)$ whose restriction to the algebra $so(n)$ is given by $\mathfrak{d} = \text{span}\{E_i \wedge E_j \mid i = 1, \dots, r, j = 1, \dots, n\}$. This implies the constraints

$$(80) \quad \langle \mathbf{w}, \text{Ad}_{g^{-1}}(E_i \wedge E_j) \rangle = \langle \mathbf{w}, e_i \wedge e_j \rangle = 0, \quad n-r+1 \leq i < j \leq n.$$

Here $\mathbf{w}(t) = g^{-1} \cdot g(t) \in so(n)$ and $e_1 = (e_{11}, \dots, e_{1n})^T, \dots, e_n = (e_{n1}, \dots, e_{nn})^T$ is the orthogonal frame of unit vectors fixed in the space and regarded in the moving frame ($E_1 = g \cdot e_1, \dots, E_n = g \cdot e_n$, where $E_1 = (1, 0, \dots, 0)^T, \dots, E_n = (0, \dots, 0, 1)^T$). They play the role of redundant coordinates on $SO(n)$.

The system is described by the kinematic Poisson equations

$$(81) \quad \dot{e}_i = -\mathbf{w}e_i, \quad i = 1, \dots, n,$$

together with the Euler-Poincaré equations with indefinite multipliers λ_{pq}

$$(82) \quad \frac{d}{dt}(\mathcal{I}\mathbf{w}) = [\mathcal{I}\mathbf{w}, \mathbf{w}] + \sum_{n-r+1 \leq p < q \leq n} \lambda_{pq} e_p \wedge e_q.$$

Since for $n = 3, r = 2$ the above system represents *Veselova problem*, we refer to $(SO(n), L_{\mathcal{I}}, \mathcal{D}_r)$ as a *generalized Veselova system* (see Fedorov and Kozlov [14]).

The Lagrangian $L_{\mathcal{I}}$ and the distribution \mathcal{D}_r are invariant with respect to the left $SO(n-r)$ -action, where $SO(n-r)$ is the subgroup of $SO(n)$, rotations that leave E_1, \dots, E_r invariant. Moreover, the distribution \mathcal{D}_r can be seen as a principal connection of the bundle

$$\begin{array}{ccc} SO(n-r) & \longrightarrow & SO(n) \\ & & \downarrow \\ & & V_{n,r} = SO(n)/SO(n-r). \end{array}$$

As a result, the system can naturally be regarded as a Chaplygin system and dynamics is reducible to the Stiefel variety $V_{n,r}$. The points of the Stiefel variety can be seen as matrices $\mathcal{X} = (e_1, \dots, e_r)$ (positions of the r -frame given by vectors (e_1, \dots, e_r)). So, the tangent bundle $TV(r, n)$ is the set of pairs $(\mathcal{X}, \dot{\mathcal{X}})$ of $n \times r$ matrices subject to the constraints

$$(83) \quad \mathcal{X}^T \mathcal{X} = \mathbf{I}_r, \quad \mathcal{X}^T \dot{\mathcal{X}} + \dot{\mathcal{X}}^T \mathcal{X} = 0.$$

The reduced Lagrangian takes the form $L_{red}(\mathcal{X}, \dot{\mathcal{X}}) = -\frac{1}{4}\text{tr}(\mathcal{I}\Phi_r\Phi_r)$ (see [16]), where the tangent bundle momentum mapping $\Phi_r : TV_{n,r} \rightarrow so(n) \cong so(n)^*$ is given by

$$\begin{aligned}\Phi_r(\mathcal{X}, \dot{\mathcal{X}}) &= \mathcal{X}\dot{\mathcal{X}}^T - \dot{\mathcal{X}}\mathcal{X}^T + \frac{1}{2}\mathcal{X}[\mathcal{X}^T\dot{\mathcal{X}} - \dot{\mathcal{X}}^T\mathcal{X}]\mathcal{X}^T \\ &= e_1 \wedge \dot{e}_1 + \cdots + e_r \wedge \dot{e}_r + \frac{1}{2} \sum_{1 \leq \alpha < \beta \leq r} [(e_\alpha, \dot{e}_\beta) - (\dot{e}_\alpha, e_\beta)] e_\alpha \wedge e_\beta.\end{aligned}$$

Introduce the $n \times r$ momentum matrix

$$(84) \quad \mathcal{P}_{is} = \partial L_{red}(\mathcal{X}, \dot{\mathcal{X}})/\partial \dot{\mathcal{X}}_{is}.$$

Since the Lagrangian is degenerate in the redundant velocities $\dot{\mathcal{X}}_{is}$, from this relation one cannot express $\dot{\mathcal{X}}$ in terms of $(\mathcal{X}, \mathcal{P})$ uniquely. On the other hand, the cotangent bundle $T^*V(r, n)$ can be realized as the set of pairs $(\mathcal{X}, \mathcal{P})$ satisfying the constraints

$$(85) \quad \mathcal{X}^T\mathcal{X} = \mathbf{I}_r, \quad \mathcal{X}^T\mathcal{P} + \mathcal{P}^T\mathcal{X} = 0.$$

Under the conditions (83), (85), the relation (84) can be uniquely inverted, i.e., one gets $\dot{\mathcal{X}} = \dot{\mathcal{X}}(\mathcal{X}, \mathcal{P})$. Then we have (see Theorem 5.4 in [16]):

Theorem 7. ([16]) *The $SO(n-r)$ -reduction of the Veselova problem (80), (81), (82) is the restriction to $T^*V(r, n)$ of the following system on the space $(\mathcal{X}, \mathcal{P})$:*

$$(86) \quad \dot{\mathcal{X}} = -\Phi_r(\mathcal{X}, \mathcal{P})\mathcal{X}, \quad \dot{\mathcal{P}} = -\Phi_r(\mathcal{X}, \mathcal{P})\mathcal{P},$$

where $\Phi_r(\mathcal{X}, \mathcal{P}) = \Phi_r(\mathcal{X}, \dot{\mathcal{X}}(\mathcal{X}, \mathcal{P}))$.

Remark 6. Here we use the opportunity to mention one correction to [16]: in the equation (5.21) the momentum mapping Φ^* should read $\Phi^* = \mathcal{I}\omega|_{\mathcal{D}_r} = \mathcal{X}\mathcal{P}^T - \mathcal{P}\mathcal{X}^T$. This equation was used only in the proof of Theorem 5.4 [16]. The statement of the theorem itself remains to be correct.

In particular, for $r = 1$, the Veselova problem is reducible to T^*S^{n-1} .

Let, as above, $A = \text{diag}(a_1, \dots, a_n)$ and denote $\gamma = \mathcal{X} = e_1$, $p = \mathcal{P}$. Taking the special inertia operator defined by

$$(87) \quad \mathcal{I}(E_i \wedge E_j) = a_i^{-1}a_j^{-1} E_i \wedge E_j, \quad 1 \leq i < j \leq n,$$

we have $\Phi_1 = \gamma \wedge \dot{\gamma} = \gamma \wedge Ap/(\gamma, A^{-1}\gamma)$ and the reduced system (86) becomes (here we replaced the matrix A from [16, 20] by A^{-1})

$$(88) \quad \dot{\gamma} = -\Phi_1\gamma = \frac{1}{(\gamma, A^{-1}\gamma)}(-(p, A\gamma)\gamma + (\gamma, \gamma)Ap),$$

$$(89) \quad \dot{p} = -\Phi_1 p = \frac{1}{(\gamma, A^{-1}\gamma)}((p, Ap)\gamma - (p, \gamma)Ap).$$

Furthermore, as it follows from [16, 20], under the time substitution (64) and the change of momenta (65) the reduced system transforms to a Hamiltonian system describing an integrable geodesic flow on T^*S^{n-1} with the Hamiltonian

$$(90) \quad \mathcal{H}(\gamma, \tilde{p}) = \frac{D^2}{2}(Ap, \tilde{p}).$$

Remark 7. The reduced Veselova system (88), (89) is trajectory equivalent to the geodesic flow on the ellipsoid $E^{n-1} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid (x, Ax) = 1\}$: the geodesic lines $x(t)$ of the ellipsoid, after the Gauss mapping $\gamma(t) = Ax(t)/|A\gamma(t)|$ and a time rescaling, become solutions of the reduced Veselova system, and vice versa (see [20]).

5.3. Integrability of the Reduced Chaplygin Sphere Problem. Let us suppose $a_i \neq a_j$, $i \neq j$. As in the three-dimensional case [12], we have

Theorem 8. (i) *The geodesic flow (69), (70) is completely integrable on T^*S^{n-1} .*
(ii) *The zero $SO(n-1)$ -momentum reduced multidimensional nonholonomic Chaplygin sphere problem (53), (54) defined by inertia operator (49) and the reduced Veselova problem (88), (89) defined by inertia operator (87) have the same invariant toric foliation of T^*S^{n-1} .*
(iii) *Let \mathbf{T} be a regular, $(n-1)$ -dimensional invariant torus. Then there exist angle coordinates $\varphi_1, \dots, \varphi_{n-1}$ on \mathbf{T} in which both problems simultaneously take the form*

$$\dot{\varphi}_1 = \frac{\omega_1^i}{D\sqrt{(A^{-1}\gamma, \gamma)}}, \dots, \dot{\varphi}_{n-1} = \frac{\omega_{n-1}^i}{D\sqrt{(A^{-1}\gamma, \gamma)}}$$

with frequencies $\omega_1^i, \dots, \omega_{n-1}^i$, $i = 1, 2$, respectively.

Proof. In what follows, we restrict our considerations to T^*S^{n-1} . The momentum integral (43) in variables (γ, \tilde{p}) becomes

$$(91) \quad K(\gamma, \tilde{p}) = D^2(A^{-1}\gamma, \gamma)(\tilde{p}, \tilde{p})$$

and the Hamiltonian (66) can be written in the form

$$H(\gamma, \tilde{p}) = \frac{1}{2D}K(\gamma, \tilde{p}) - \frac{1}{D^2}\mathcal{H}(\gamma, \tilde{p}).$$

Since (91) is the integral of the geodesic flow (69), (70), on T^*S^{n-1} we have

$$\{H, K\}_d = \{H, \mathcal{H}\}_d = \{\mathcal{H}, K\}_d = 0.$$

Consider the spherconical coordinates $(\lambda_1, \dots, \lambda_{n-1})$ ($a_1 < \lambda_1 < a_2 < \dots < \lambda_{n-1} < a_n$) on S^{n-1} defined by the relations

$$(92) \quad \gamma_i^2 = \frac{(a_i - \lambda_1) \cdots (a_i - \lambda_{n-1})}{\prod_{j \neq i} (a_i - a_j)}, \quad i = 1, \dots, n$$

(see [26]). Let $(\mu_1, \dots, \mu_{n-1})$ be the canonical momenta on the cotangent bundle with respect to the form (45)

$$\tilde{\omega} = d\tilde{p}_1 \wedge d\gamma_1 + \cdots + d\tilde{p}_n \wedge d\gamma_n|_{T^*S^{n-1}} = d\mu_1 \wedge d\lambda_1 + \cdots + d\mu_{n-1} \wedge d\lambda_{n-1}.$$

Then, according to [26], [6] and [16], respectively, we have:

$$\begin{aligned} (\tilde{p}, \tilde{p}) &= -4 \sum_{k=1}^{n-1} \frac{(\lambda_k - a_1) \cdots (\lambda_k - a_n)}{\prod_{s \neq k} (\lambda_k - \lambda_s)} \mu_k^2, \\ (\gamma, A^{-1}\gamma) &= \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{a_1 a_2 \cdots a_n}, \\ (A\tilde{p}, \tilde{p}) &= -4 \sum_{k=1}^{n-1} \frac{(\lambda_k - a_1) \cdots (\lambda_k - a_n) \lambda_k}{\prod_{s \neq k} (\lambda_k - \lambda_s)} \mu_k^2. \end{aligned}$$

Therefore, the Hamiltonian (90) has the Stäckel form in spherconical variables and the geodesic flow on T^*S^{n-1} determined by \mathcal{H} is completely integrable (see [16]). We have Poisson commuting, quadratic in momenta integrals F_1, \dots, F_{n-1} (e.g., see [1]). One can prove that functions F_i commute with H using the direct calculations in elliptic coordinates.

Alternatively, note the geodesic flow of \mathcal{H} , over a generic invariant torus \mathbf{T} (level set of F_1, \dots, F_{n-1}) is quasi-periodic with non-resonant frequencies (for example this follows from Remark 7). Thus, since $\{\mathcal{H}, K\}_d = 0$ and the integral trajectories are dense on \mathbf{T} , K is also constant along the Hamiltonian flows of F_i over \mathbf{T} . Since we deal with analytic functions, we get that K is in involution with F_1, \dots, F_{n-1} on the whole T^*S^{n-1} (K is the analogue of the classical Joachimsthal's integral of the geodesic flow on the ellipsoid E^{n-1} [26]). Further, the Hamiltonian H , as a linear combination of K and \mathcal{H} , Poisson commutes with F_i as well. Whence, the system (69), (70) is completely integrable on T^*S^{n-1} .

The last assertion of the Theorem follows from the Liouville-Arnold theorem [1] and the fact that the systems transform to a Hamiltonian form after the same time reparametrization (64). \square

The system is integrable even if not all a_i are distinct. For any pair of equal parameters $a_i = a_j$, the geodesic flow (69), (70) has the additional linear integral $f_{ij} = \gamma_i \tilde{p}_j - \gamma_j \tilde{p}_i$. For example, let $n = 4$ and $a_1 = a_2 \neq a_3 = a_4$. Then the complete set of commuting integrals is f_{12} , f_{34} and H . If we have at least three equal parameters, the system is integrable according to the non-commutative version of the Liouville theorem.

Remark 8. Note that the operators (49) and (87) are related via

$$D\mathcal{I} = \mathbf{I} + DI^{-1} \iff I = D(D\mathcal{I} - \mathbf{I})^{-1}.$$

In order to reobtain Fedorov's correspondence (79) for $n = 3$ and $f_1 = F_1 = 0$, instead of (87) one should consider the inertia operator multiplied by D .

5.4. Lagrange Case. Consider the Lagrange case (52). Due to the additional $SO(n-1)$ -symmetry, the geodesic flow (69), (70) has the integrals f_{ij} , $1 \leq i < j \leq n-1$. Thus, in the original coordinates we get integrals

$$(93) \quad F_{ij} = (\gamma, A^{-1}\gamma)(\gamma_i p_j - \gamma_j p_i)^2, \quad 1 \leq i < j \leq n-1.$$

In this case we do not need Hamiltonization to integrate the reduced system, it is already integrable according to the *Euler-Jacobi theorem*. Since the generic invariant manifolds given by H and integrals (93) are two-dimensional and the system has an invariant measure we have [1]:

Theorem 9. *The Lagrange case of the reduced Chaplygin system (53), (54) is solvable by quadratures; compact regular invariant manifolds given by functions (93) and (66) are two-dimensional tori.*

Remark 9. Although the Lagrangian (12) is additionally invariant with respect to the right $SO(n-1)$ -action, the integrals (93) are not Noether's integrals. The reason is that the associated vector fields do not satisfy constraints (13). For $n=3$ and $I_1 = I_2$, the corresponding integral of the system (30) is $F = k_3^2 - D(\gamma, (I + D\mathbf{I})^{-1}\gamma)k_3^2$.

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